V. Acknowledgment

The author is grateful to his colleagues at the National Research Council of Canada, in particular to J. K. Pulfer, for helpful advice and criticism given in the course of preparing this paper.

References

[1] Cerrillo, M. V., Transient phenomena in waveguides, Tech Rept 33, MIT Research Lab. of Electronics, Cambridge, Mass., Jan 1948.
[2] Cohn, G. I., Electromagnetic transients in waveguide, *Proc. NEC*, vol 8, Oct 1952, pp 284–295.
[3] Cotte, M., Propagation d'une Perturbation dans un Guide Electrique, *Ann. Télécommun*, vol 1, Mar–Apr 1946, pp 49–52.
[4] ——, Propagation d'une Impulsion sur un Guide D'Ondes, *Onde elect.*, vol 34, Feb 1954, pp 143–146.
[5] Pearson, J. D., The transient motion of sound waves in tubes, *Quart. J. Mech. and Appl. Math.*, vol 6, pt 3, 1953, pp 313–335.
[6] Poincelot, P., Propagation d'un Signal Le Long d'un Guide D'Ondes, *Ann. Télécommun.*, vol 9, Nov 1954, pp 315–317.
[7] Karbowiak, A. E., Propagation of transients in waveguides, *Proc. IEE (London)*, vol 104, pt C, Sep 1957, pp 339–348.
[8] Kovtun, A. A., Transient processes in waveguide, *Radio Eng. Electron.*, vol 3, no. 5, 1958, pp 103–122.
[9] Gajewski, R., Influence of wall losses on pulse propagation in waveguides, *J. Appl. Phys.*, vol 29, Jan 1958, pp 22–24.
[10] Elliot, R. S., Pulse waveform degradation due to dispersion in waveguide, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-5, Oct 1957, pp 254–257.
[11] Knop, C. M., and G. I. Cohn, Comments on "pulse waveform degradation due to dispersion in waveguide," *IEEE Trans. on Microwave Theory and Techniques (Correspondence)*, vol MTT-11, Sep 1963, pp 445–447.
[12] Wanselow, R. D., Rectangular pulse distortion due to a nonlinear complex transmission propagation constant, *J. Franklin Inst.*, vol 274, Sep 1962, pp 178–184.
[13] Forrer, M. P., Analysis of millimicrosecond RF pulse transmission, *Proc. IRE*, vol 46, Nov 1958, pp 1830–1835.
[14] Winter, D. F., E. A. Guillemin, L. J. Chu, and M. V. Cerrillo, Transient phenomena in waveguides, Quart. Prog. Rept, MIT Research Lab. of Electronics, Cambridge, Mass., Apr 1947.
[15] Beck, A. C., Microwave testing with millimicrosecond pulses, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-2, Apr 1954, pp 93–99.
[16] ——, Measurement techniques for multimode waveguides, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-3, Apr 1955, pp 35–41.
[17] Saxton, W. A., and H. J. Schmitt, Transients in large waveguide, *Proc. IEEE (Correspondence)*, vol 51, Feb 1963, pp 405–406.
[18] Anderson, D. V., and C. Barnes, The dispersion of a pulse propagated through a cylindrical tube, *J. Acoust. Soc. Am.*, vol 25, May 1953, pp 525–528.
[19] Proud, J. M., P. Tamarkin, and E. T. Kornhauser, Propagation of sound pulses in a dispersive medium, *J. Acoust. Soc. Am.*, vol 28, Jan 1956, pp 80–85.
[20] Walther, K., Pulse compression in acoustic waveguide, *J. Acoust. Soc. Am.*, vol 33, May 1961, pp 681–686.
[21] De Bruijn, N. C., *Asymptotic Methods in Analysis.* Amsterdam, Holland: North-Holland Publishing Co., 1958, ch 5.
[22] Papoulis, A., *The Fourier Integral and Its Applications.* New York: McGraw-Hill, 1962, sec 7-7.
[23] Haggarty, R. D., A method of signal design employing saddle-point integration, Rept 41G0009, MIT Lincoln Lab., Lexington, Mass., May 1961.
[24] Tverskoi, V. I., Use of delay system with dispersion of phase velocity for the analysis of spectra of individual radio signals, *Radio Eng. Electron.*, vol 4, no 9, 1959, pp 242–246.
[25] Wind, M., and H. Rapaport, *Handbook of Microwave Measurements*, vols I and II. Brooklyn, N. Y.: Polytech. Inst. Brooklyn Microwave Res. Inst., 1954–1955, sec 14.

The Solution of Guided Waves in Inhomogeneous Anisotropic Media by Perturbation and Variational Methods

G. J. GABRIEL, STUDENT MEMBER, IEEE, AND M. E. BRODWIN, MEMBER, IEEE

***Abstract*—The Schroedinger perturbation theory is extended to the boundary value problems of guided electromagnetic waves in cylindrical structures containing inhomogeneous, anisotropic, dissipative media. A general variational principle, which reduces to existing restricted forms valid for nondissipative media, is also formulated. These approximation methods evolve in a unified manner from the eigenvalue formulation of Maxwell's equations wherein the wave numbers are the eigenvalues of a linear operator. The properties of the media are restricted only by the requirement that they be independent of the axial coordinate. Perturbation of the backward wave is considered and the condition for nonreciprocal waveguides is stated. Modification of the perturbation method for application to gyrotropic media is outlined and it is indicated that convergence of the perturbation terms is improved in those media, such as the plasma and semiconductor, which permit a Taylor expansion of the susceptibility tensor in powers of the external field. Two specific examples, whose exact solutions are known, are included to illustrate the application.**

Manuscript received October 23, 1964; revised February 4, 1965. This work was supported by the Advanced Research Projects Agency through the Northwestern University Materials Research Center.

The authors are with the Dept. of Electrical Engineering, Northwestern University, Evanston, Ill.

I. Introduction

THE PROPAGATION of guided electromagnetic waves in cylindrical structures containing anisotropic, inhomogeneous media poses formidable boundary value problems, even under simplifying conditions. In recent years, materials which display induced anisotropy, namely the gyrotropic media, have received considerable attention. Because of the special form of the susceptibility tensors of gyrotropic media, when the external magnetic field is oriented along one of the coordinate axes, exact analytical treatment of the boundary value problem in a limited number of geometric configurations is possible [1]–[4]. The resulting eigenvalue equations for the allowed wave numbers, however, are often too complicated to permit explicit expression of the roots. Numerical or approximate analytical methods are then used for determining these roots.

When the media are inherently inhomogeneous and anisotropic, without dependence on an external influence such as the magnetic field, the form of the tensors is not restricted; and when the media are dissipative, the tensors are necessarily non-Hermitian. The mathematical complexity of the problem then increases substantially. In this general case, an approximate analysis of the boundary value problem is simpler than exact solution, if not in fact the only resort. Even in the special case of gyrotropic media, under certain circumstances and especially when the media are dissipative such as the semiconductor, an approximate analysis may yield sufficiently accurate results to diminish the merit of the exact solution.

It is possible to treat inhomogeneity and anisotropy, induced and inherent, by viewing the presence of the media as a perturbation of the empty waveguide. The formulation of the Maxwell equations as an eigenvalue problem in operator notation, initiated by Bresler, Joshi, and Marcuvitz [5], [6], enables the extension of the scalar Schroedinger perturbation theory to problems of guided waves in cylindrical structures, and permits the application of the powerful techniques of the spectral theory of operators. A general variational approximation, which reduces to the restricted form given by Berk [7] for media possessing Hermitian tensor parameters, also follows from the eigenvalue formulation. In the present perturbation and variational approximations, no restrictions are imposed except that the permeability and permittivity tensor elements be independent of the axial coordinate. These elements may, however, be functions of the coordinates in the transverse plane.

Section II of this paper is concerned with the extension of the scalar perturbation theory to problems of guided waves. In Section III the general variational method is considered. Section IV consists of specific examples to illustrate the application of the perturbation theory. Under consideration are cylindrical waveguides, whose axes coincide with the z axis of an appropriate cylindric coordinate system (u_1, u_2, z). The waveguides are assumed to contain linear, inhomogeneous, anisotropic, dissipative media characterized by the permeability and permittivity tensors (dyadics) $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ which are independent of z. The walls of the waveguides are perfect conductors, and all fields are assumed to depend on z and the time t through the factor $\exp i(\kappa z - \omega t)$.

II. Perturbation Theory

The underlying principle of the perturbation theory is that the presence of material media may be regarded as a perturbation of the empty waveguide, provided the media cause sufficiently small effects. The theory evolves in a unified manner from the formulation of the Maxwell equations as an eigenvalue problem, under the assumption of exponential z and t dependence, wherein the wave numbers are the eigenvalues of a linear operator. In this formulation [5], the electromagnetic field is characterized by the six-vector function

$$\Phi_\alpha(u_1, u_2) = \begin{bmatrix} E_\alpha \\ iH_\alpha \end{bmatrix} \tag{1}$$

in terms of which the Maxwell equations take the form

$$(\mathcal{L} - \kappa_\alpha \boldsymbol{\Gamma})\Phi_\alpha = 0 \tag{2}$$

where

$$\mathcal{L} = \begin{bmatrix} \omega\boldsymbol{\varepsilon} & -\nabla_t \times I \\ -\nabla_t \times I & \omega\boldsymbol{\mu} \end{bmatrix}$$

and

$$\boldsymbol{\Gamma} = \begin{bmatrix} 0 & i\hat{z} \times I \\ i\hat{z} \times I & 0 \end{bmatrix}$$

and I is the unit dyad, $\hat{z}$ the unit vector along the z axis, and ∇_t involves only the transverse coordinates. Matrix rules of operation are implied, with the provision that the dot product is used for the product of dyadics and vectors. The domain of $\mathcal{L}$ is specified by the boundary conditions imposed on Φ_α at the guide walls.

The inhomogeneity and anisotropy of the medium filling the guide are embodied in the operator $\mathcal{L}$. Since these properties are conveyed as well by the magnetic and electric susceptibilities χ_m and χ_e, $\mathcal{L}$ can always be resolved into

$$\mathcal{L} = \mathcal{L}_0 + L \tag{3}$$

where $\mathcal{L}_0$ is the operator corresponding to free space; i.e., $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ in (2) are replaced by $\mu_0 I$ and $\epsilon_0 I$, and

$$L = \omega\begin{bmatrix} \epsilon_0\chi_e & 0 \\ 0 & \mu_0\chi_m \end{bmatrix}. \tag{4}$$

It is important to note that L, which is the perturbation operator, is restricted only by the requirement that it be independent of z. In addition to its dependence on the transverse coordinates (u_1, u_2), L may also be a function of a parameter ν which accounts for the possible dependence of the material properties on factors foreign to the microwave field. For example, in gyrotropic media, the external magnetic intensity is the parameter which induces the anisotropy.

With the viewpoint that L is a perturbation on $\mathcal{L}_0$, the eigenvectors and eigenvalues of $\mathcal{L}$ differ from those of $\mathcal{L}_0$ by small additive terms. Accordingly, we introduce the "dummy" parameter λ, which will aid in the grouping of the perturbation terms and which is equated to unity in the final results; thus

$$\mathcal{L}(u_1, u_2, \nu) = \mathcal{L}_0 + \lambda L(u_1, u_2, \nu) \tag{5}$$

$$\Phi_\alpha(u_1, u_2, \nu) = \Phi_{\alpha 0}(u_1, u_2) + \sum_{n=1} \lambda^n \Phi_{\alpha n}(u_1, u_2, \nu) \tag{6}$$

$$\kappa_\alpha(\nu) = \kappa_{\alpha 0} + \sum_{n=1} \lambda^n \kappa_{\alpha n}(\nu) \tag{7}$$

where $\Phi_{\alpha 0}$ and $\kappa_{\alpha 0}$ are the zero-order eigenvectors and eigenvalues of the unperturbed waveguide. In this notation, the first subscript identifies the eigenvectors while the second subscript designates the order of the perturbation. The conditions under which the perturbation analysis is applicable and the rapidity of convergence of the series (6) and (7) can not be resolved in general terms. These questions are contingent upon the extent to which the media alter the homogeneous, isotropic structure and, hence, can be settled only when the conditions in a particular configuration are known.

The recurrence equations which the perturbation fields must satisfy are obtained by the usual procedure of substituting (5), (6), and (7) into (2) and grouping the coefficients of like powers of λ. The first three of these equations are

$$(\mathcal{L}_0 - \kappa_{\alpha 0}\mathbf{\Gamma})\Phi_{\alpha 0} = 0 \tag{8}$$

$$(\mathcal{L}_0 - \kappa_{\alpha 0}\mathbf{\Gamma})\Phi_{\alpha 1} = -(L - \kappa_{\alpha 1}\mathbf{\Gamma})\Phi_{\alpha 0} \tag{9}$$

$$(\mathcal{L}_0 - \kappa_{\alpha 0}\mathbf{\Gamma})\Phi_{\alpha 2} = -(L - \kappa_{\alpha 1}\mathbf{\Gamma})\Phi_{\alpha 1} + \kappa_{\alpha 2}\mathbf{\Gamma}\Phi_{\alpha 0}. \tag{10}$$

Once a suitable orthonormal zero-order set is constructed, $\Phi_{\alpha n}$ and $\kappa_{\alpha n}$ can be determined.

A. The Bi-orthogonal Set

In this section, we briefly review the orthogonality relations of Bresler, Joshi, and Marcuvitz [5] to the extent needed for the evaluation of the perturbation terms and for use in the variational approximation. Throughout this paper, the symmetric scalar product of two six-vectors is defined by

$$\langle \Phi_\alpha | \Phi_\beta \rangle = \int (E_\alpha \cdot E_\beta + iH_\alpha \cdot iH_\beta) da \tag{11}$$

where the integration is over the cross section of the guide.

If Φ_α^T is the eigenvector of the adjoint operator $\mathcal{L}^T$ such that

$$(\mathcal{L}^T + \kappa_\alpha \mathbf{\Gamma})\Phi_\alpha^T = 0 \tag{12}$$

then the sets $\{\Phi_\alpha^T\}$ and $\{\Phi_\beta\}$ comprise a bi-orthogonal set with respect to the "weight" operator $\mathbf{\Gamma}$; that is,

$$\langle \Phi_\alpha^T | \mathbf{\Gamma}\Phi_\beta \rangle = N_\alpha \delta_{\alpha\beta} \tag{13}$$

where N_α is the normalization constant and $\delta_{\alpha\beta}$ is the Kronecker delta. The adjoint operator is defined by the relation

$$\langle \mathcal{L}^T \Phi^T | \Phi \rangle = \langle \Phi^T | \mathcal{L}\Phi \rangle$$

and it is obtained by replacement of $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ in (2) with $\boldsymbol{\mu}^T$ and $\boldsymbol{\varepsilon}^T$, these being the transposed dyadics. The analysis has been restricted to guides with perfectly conducting walls so that the domains of $\mathcal{L}$ and $\mathcal{L}^T$ are identical. The possibility of generalized impedance walls is discussed in the cited references.

It will be noticed that for every eigenvalue κ_α associated with Φ_α, an eigenvector of $\mathcal{L}$, there is an eigenvalue $-\kappa_\alpha$ associated with Φ_α^T, an eigenvector of $\mathcal{L}^T$. A negative eigenvalue implies a wave traveling in the opposite direction. Hence, Φ_α^T is identified as the backward traveling wave in the "adjoint waveguide," this being a waveguide filled with a medium characterized by the transposed dyadic parameters [8]. In general, the "adjoint function" Φ_α^T has no physical meaning, and it appears in the problem only as a subsidiary set in terms of which the bi-orthogonality condition is defined. In gyrotropic media, however, with the external magnetic field oriented along a coordinate axis, the transposition of the dyadics (tensors) is physically meaningful as this amounts to a reversal of the magnetic field.

It can be shown that a sufficient condition for $-\kappa_\alpha$ to be an eigenvalue of $\mathcal{L}$, when κ_α is an eigenvalue, is that the operator be symmetric. In the special case when $\mathcal{L} = \mathcal{L}_0$, the symmetric operator of the conventional waveguide, both $\pm\kappa_\alpha$ are eigenvalues. Then Φ_α reduces to $\Phi_{\alpha 0}$ and Φ_α^T to $\Phi_{-\alpha 0}$ which are the forward and backward traveling waves, respectively, and are the usual TE- and TM-type modal functions. The bi-orthogonality condition then becomes

$$\langle \Phi_{\mp\alpha 0} | \mathbf{\Gamma}\Phi_{\pm\beta 0} \rangle = \pm N_\alpha \delta_{\alpha\beta}. \tag{14}$$

It is important to note also that

$$\langle \Phi_{\pm\alpha 0} | \mathbf{\Gamma}\Phi_{\pm\alpha 0} \rangle = 0. \tag{15}$$

By "conventional" waveguide here is meant the empty waveguide or the waveguide completely filled with a linear, homogeneous, isotropic, dissipative medium.

Since both $\pm\kappa_\alpha$ are eigenvalues of $\mathcal{L}_0$, the complete spectral functions in a transverse plane are $\Phi_{\alpha 0}$ and $\Phi_{-\alpha 0}$. The usual expansion of fields in cylindrical waveguides takes the following form in the six-vector formalism

$$\Phi(u_1, u_2, z) = \sum_\alpha (a_\alpha e^{i\kappa_{\alpha 0} z}\Phi_{\alpha 0} - a_{-\alpha} e^{-i\kappa_{\alpha 0} z}\Phi_{-\alpha 0}). \tag{16}$$

The coefficients are given by

$$a_\alpha = e^{-i\kappa_{\alpha 0} z_0}\langle \Phi_{-\alpha 0} | \mathbf{\Gamma}\Phi \rangle$$
$$a_{-\alpha} = e^{i\kappa_{\alpha 0} z_0}\langle \Phi_{\alpha 0} | \mathbf{\Gamma}\Phi \rangle$$

where Φ is evaluated at z_0 and the set has been assumed to be normalized. In the summation (16) a special index has not been used to distinguish degenerate eigenvectors. When a κ_β is n-fold degenerate, the index α is understood to take successively n values for the same κ_β. We are now in position to evaluate the perturbation terms.

B. First- and Second-Order Nondegenerate Perturbation

Only the first- and second-order terms for a nondegenerate wave modal function are considered. The extension to degenerate functions is similar to the methods of quantum mechanical problems and is not included here. In the unperturbed state, the waveguide is assumed to sustain only the forward traveling mode $\Phi_{\alpha 0}$. Because of the completeness of the zero-order set, the first-order field may be expanded in the series

$$\Phi_{\alpha 1} = \sum_\beta (a_{\beta 1}\Phi_{\beta 0} - a_{-\beta 1}\Phi_{-\beta 0}). \tag{17}$$

When this is substituted in (9) and the bi-orthogonality relation (14) is used in conjunction with the fact that all the $\Phi_{\alpha n}$ are in the domain of $\mathfrak{L}_0$, the eigenvalue perturbation and the coefficients $a_{\pm\beta 1}$ are determined to be

$$\kappa_{\alpha 1} = L_{-\alpha,\alpha} \tag{18}$$

$$a_{\beta 1} = \frac{L_{-\beta,\alpha}}{\kappa_{\alpha 0} - \kappa_{\beta 0}}, \qquad \beta \neq \alpha \tag{19}$$

$$a_{-\beta 1} = \frac{L_{\beta,\alpha}}{\kappa_{\alpha 0} + \kappa_{\beta 0}}, \qquad \text{all } \beta \tag{20}$$

where

$$L_{i,j} = \langle \Phi_{i0} \mid L\Phi_{j0} \rangle.$$

The coefficient $a_{\alpha 1}$ is equated to zero by requiring that the perturbed eigenvector be normalized to first order in the parameter [9]. Equation (18) is in agreement with the viewpoint that, to first order in the nondissipative medium, the increase in the wave number is proportional to the increase in the stored electromagnetic energy.

With an expansion similar to (17) for the second-order term, the following is obtained:

$$\kappa_{\alpha 2} = \sum_\beta{}' \frac{L_{-\alpha,\beta}L_{-\beta,\alpha}}{\kappa_{\alpha 0} - \kappa_{\beta 0}} - \sum_\beta \frac{L_{-\alpha,-\beta}L_{\beta,\alpha}}{\kappa_{\alpha 0} + \kappa_{\beta 0}} \tag{21}$$

$$a_{\gamma 2} = \sum_\beta{}' \frac{L_{-\gamma,\beta}L_{-\beta,\alpha}}{(\kappa_{\alpha 0} - \kappa_{\beta 0})(\kappa_{\alpha 0} - \kappa_{\gamma 0})} - \sum_\beta \frac{L_{-\gamma,-\beta}L_{\beta,\alpha}}{(\kappa_{\alpha 0} + \kappa_{\beta 0})(\kappa_{\alpha 0} - \kappa_{\gamma 0})} - \frac{L_{-\gamma,\alpha}L_{-\alpha,\alpha}}{(\kappa_{\alpha 0} - \kappa_{\gamma 0})^2} \tag{22}$$

$$a_{-\gamma 2} = \sum_\beta{}' \frac{L_{\gamma,\beta}L_{-\beta,\alpha}}{(\kappa_{\alpha 0} - \kappa_{\beta 0})(\kappa_{\alpha 0} + \kappa_{\gamma 0})} - \sum_\beta \frac{L_{\gamma,-\beta}L_{\beta,\alpha}}{(\kappa_{\alpha 0} + \kappa_{\beta 0})(\kappa_{\alpha 0} + \kappa_{\gamma 0})} - \frac{L_{\gamma,\alpha}L_{-\alpha,\alpha}}{(\kappa_{\alpha 0} + \kappa_{\gamma 0})^2} \tag{23}$$

where the prime on the summation sign denotes exclusion of the term $\beta = \alpha$.

The perturbation fields have been expanded in the zero-order eigenvectors which are similar, but not identical, to the modal functions of the conventional waveguide. The zero-order eigenvectors do not constitute modes in the usual sense. In the conventional guide, each modal solution propagates with a distinct wave number independently of all the others. Here, the αth perturbed eigenvector is a superposition of the field distributions of the six components of the modal functions only insofar as the functional dependence on the transverse coordinates is concerned.

C. Perturbation of the Backward Wave

Thus far only the perturbation of the forward traveling wave has been considered. When a backward wave is present the treatment is not altered markedly. Let the backward wave be

$$\Phi_{-\alpha} = \Phi_{-\alpha}(u_1, u_2)e^{-i\kappa'_\alpha z}.$$

Any field that exists in a waveguide must be an eigenvector, or combination of eigenvectors, of the operator $\mathfrak{L}$ which properly describes that particular waveguide. Thus, $\Phi_{-\alpha}$ must be a solution of

$$(\mathfrak{L} + \kappa_\alpha'\mathbf{\Gamma})\Phi_{-\alpha} = 0, \tag{24}$$

which differs from (2) in the sign of the eigenvalue. When the perturbation process is applied to (24), expressions identical to those for the forward wave are obtained except that the sign of the index α is changed, it being understood that $\kappa_{-\alpha 0} = -\kappa_{\alpha 0}$.

In general, the wave numbers κ_α' and κ_α are not equal. Waveguides which display this property are commonly described as nonreciprocal. A necessary condition for the nonreciprocity is that $\mathfrak{L}$ be asymmetrical. The asymmetry, however, is not sufficient since the equality $\kappa_\alpha' = \kappa_\alpha$ may materialize under special circumstances even when $\mathfrak{L}$ and $\mathfrak{L}^T$ are not equal.

It is interesting to examine the first-order perturbation of κ_α'. This is given by

$$\kappa_{\alpha 1}' = L_{\alpha,-\alpha} \tag{25}$$

where the signs of the indices in (18) have been changed. From the definitions of $L_{i,j}$ and the adjoint operator, we have

$$\kappa_{\alpha 1}' = L^T_{-\alpha,\alpha}. \tag{26}$$

Thus, the first-order perturbation of the backward wave number is identical to the perturbation of the forward wave number in the transposed medium. This similarity to the transposed medium, however, *does not* extend to the eigenvectors nor to the higher order perturbations of the eigenvalues. This becomes evident by noting that reversal of the sign of α in (19)–(23) is not in general equivalent to replacement of $L_{i,j}$ with $L_{i,j}{}^T$. We note also that when L is symmetric, then $\kappa_{\alpha 1}'$ reduces to $\kappa_{\alpha 1}$.

III. Variational Method

The variational method is another approximation technique which follows readily from the eigenvalue formulation of the Maxwell equations. Berk [7] has derived for the wave numbers a restricted vector variational principle, which is valid only when $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ are Hermitian, thereby excluding dissipative media. Since $\mathcal{L}$ is necessarily non-Hermitian for dissipative media, the appropriate general variational principle, which is obtained from (2) and which is similar to the scalar form [10], is given by

$$\kappa = \frac{\langle \Phi^T \mid \mathcal{L}\Phi \rangle}{\langle \Phi^T \mid \boldsymbol{\Gamma}\Phi \rangle} \tag{27}$$

where Φ^T is a solution of the transposed (adjoint) equation. The fact that κ is stationary for all Φ and Φ^T which are solutions of (2) and (12), respectively, can be verified by assuming Φ^T initially to be some Ψ and showing that the variation of κ vanishes only if

$$(\mathcal{L}^T + \kappa\boldsymbol{\Gamma})\Psi = 0. \tag{28}$$

That is, Ψ must be equal to Φ^T. Furthermore, with the aid of the general bi-orthogonality relation (13), it is not difficult to show that κ is an upper bound of the lowest eigenvalue, provided $\mathcal{L}$ is Hermitian. The proof is inconclusive when $\mathcal{L}$ is not Hermitian.

To prove that the variational expression given by Berk is contained in the general form (27), we note that when $\mathcal{L}$ is Hermitian, then $\mathcal{L}^{T*} = \mathcal{L}$ and $\kappa^* = \kappa$; and since $\boldsymbol{\Gamma}^* = -\boldsymbol{\Gamma}$, it follows that Φ^{T*} is a solution of

$$(\mathcal{L} - \kappa\boldsymbol{\Gamma})\Phi^{T*} = 0.$$

Comparison of this with (2) indicates that $\Phi^T = \Phi^*$. Hence, when $\mathcal{L}$ is Hermitian, the general expression (27) assumes the form

$$\kappa = \frac{\langle \Phi^* \mid \mathcal{L}\Phi \rangle}{\langle \Phi^* \mid \boldsymbol{\Gamma}\Phi \rangle}. \tag{29}$$

When written in terms of the components, this expression appears as

$$\kappa = \frac{\omega \int (E^* \cdot \boldsymbol{\varepsilon} \cdot E + H^* \cdot \boldsymbol{\mu} \cdot H) da + i \int (H^* \cdot \nabla_t \times E - E^* \cdot \nabla_t \times H) da}{\int (H^* \cdot \hat{z} \times E - E^* \cdot \hat{z} \times H) da} \tag{30}$$

which is the formula obtained by Berk.

In the first approximation, the variational and perturbation methods lead to identical results for the eigenvalues. This can be verified by putting (27) in the alternative form

$$\kappa = \frac{\langle \Phi^T \mid \mathcal{L}_0\Phi \rangle + \langle \Phi^T \mid L\Phi \rangle}{\langle \Phi^T \mid \boldsymbol{\Gamma}\Phi \rangle}. \tag{31}$$

As "trial" solutions, the normalized zero-order eigenvectors $\Phi_{\alpha 0}$ and $\Phi_{-\alpha 0}$ yield

$$\kappa = \kappa_{\alpha 0} + L_{-\alpha,\alpha} \tag{32}$$

which is in agreement with (18).

This perturbation formula for the wave number reduces to that given by Berk for the special case of nondissipative media having Hermitian $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ tensors. This may be shown readily by writing $L_{-\alpha,\alpha}$ in terms of the components (E, iH) and in accordance with the definition of $L_{i,j}$ in (20), and noting that $\Phi_{-\alpha 0} = \Phi_{\alpha 0}^*$ in nondissipative media.

IV. Application

In the analysis of guided electromagnetic waves in anisotropic, inhomogeneous media, it is convenient to classify problems into three categories depending on the nature of the media:

1) Media which are inherently inhomogeneous and anisotropic; $L = L(u_1, u_2)$. Examples of these are the optically active crystals and the homogeneous, isotropic media partially filling a waveguide.
2) Media which are inherently homogeneous, isotropic but which become anisotropic, and possibly inhomogeneous, by induction when subjected to influences foreign to the microwave fields and characterized by the parameter ν; $L = L(\nu)$. Examples of these are the gyromagnetic and gyroelectric media in the field of an external magnet and completely filling a waveguide.
3) Media which are inhomogeneous and anisotropic both inherently and by induction; $L = L(u_1, u_2, \nu)$. An example of these is the gyromagnetic medium partially filling a waveguide.

Problems in all three categories may be treated by the approximation methods outlined. In the perturbation analysis of the special case of induced anisotropy, however, ν may supplant λ as the perturbation parameter, provided L is a known continuous function of ν. Then the elements of L can be expanded in a Taylor series about $\nu = 0$,

$$L = \sum_{n=1} \nu^n L_n. \tag{33}$$

In this case [category 2)], the perturbation theory is most successful since rapid convergence can be assured by the simple expediency of maintaining the values of ν within a prescribed upper bound. It is understood that $\Phi_{\alpha n}$ and $\kappa_{\alpha n}$ in (6) and (7) are no longer functions of ν, and that the zero-order eigenfunctions belong to the filled waveguide but in the absence of the induced anisotropy or inhomogeneity.

Waveguides which are completely filled with a semiconductor or plasma, in the field of an external magnet, lend themselves especially to this latter mode of analysis. Here, the external magnetic field intensity is the perturbation parameter. Since expressions for the conductivity and susceptibility tensor elements as functions of the magnetic field are available, the Taylor series (33) can be obtained. This method was applied to the problem of rectangular waveguide filled with a semiconductor in a transverse magnetic field. The results, which are in preparation for publication, are in excellent agreement with experiment and are much simpler than the exact solution.

In contradistinction, the waveguide filled with a ferrite, although in the same category as the plasma and the semiconductor case, does not fit this approach very successfully for two reasons. First, both the external intensity $\boldsymbol{H}_0$ and the internal magnetization $\boldsymbol{M}_0$ appear explicitly in the expressions for the magnetic susceptibility tensor elements. Because of nonlinearity and hysteresis effects in ferrites, exact analytical relation between $\boldsymbol{H}_0$ and $\boldsymbol{M}_0$ is not known. Second, the region of the B-H characteristics that is of greatest interest in practice is that of saturation where the field intensity may be too large to be effectively regarded as a small perturbation. The evaluation of higher order terms becomes necessary, and these progressively increase in complexity. In the ferrite problems, there appears to be no advantage in the use of ν instead of λ as the parameter.

We consider now two problems, whose solutions are known, for the purpose of illustrating the application of the perturbation method. The conventional E- and H-type modal functions are employed as the zero-order orthonormal set. These functions, together with their respective normalization constants, are listed in the Appendix for reference.

A. Dielectric Slab in Rectangular Waveguide

As the first example, consider the rectangular waveguide with a homogeneous, isotropic dielectric slab symmetrically placed in the manner of Fig. 1. This configuration falls in the category of inherent inhomogeneity [category 1)] and accordingly

$$L(x) = \omega\epsilon_0\chi_e\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}; \quad \tfrac{1}{2}(x_0-\delta) \le x \le \tfrac{1}{2}(x_0+\delta). \tag{34}$$

We assume that the fundamental nondegenerate $H_{1,0}$ mode is the only one present in the absence of the dielectric and proceed to evaluate

$$\kappa_{(1,0)1} = \omega\epsilon_0\chi_e \int \boldsymbol{E}_{-(1,0)}\cdot\boldsymbol{E}_{1,0}da \tag{35}$$

where $\boldsymbol{E}_{1,0}$ is the component of the normalized eigenvector. When the appropriate function from the Appendix is introduced and the integration is performed, the following is obtained:

$$\kappa_{(1,0)1} = \frac{k_0^2\chi_e}{2\kappa_{(1,0)0}}\left[\frac{\delta}{x_0} + \frac{1}{\pi}\sin k_{1,0}\delta\right] \tag{36}$$

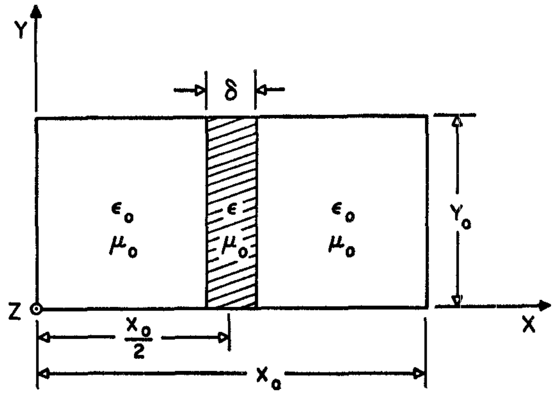

Fig. 1. Geometry of the problem of the isotropic homogeneous dielectric slab symmetrically placed in rectangular waveguide. An example of category 1).

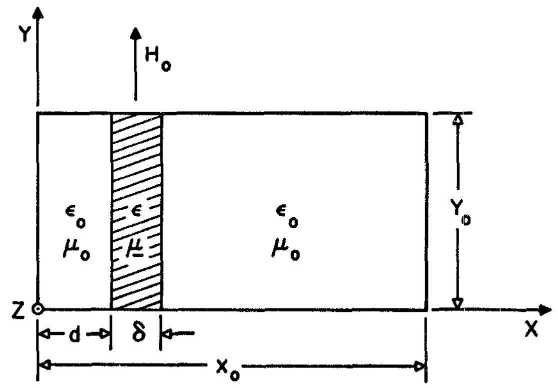

Fig. 2. Geometry of the ferrite phase shifter in rectangular waveguide. An example of category 3).

where $k_{1,0}$ is the cutoff wave number and $k_0 = \omega\sqrt{\mu_0\epsilon_0}$. This result, which was obtained by Berk using the variational method, has been shown by numerical examples to be in excellent agreement with the exact solution [7]. In that earlier variational approximation, χ_e was necessarily assumed to be real, but we assert that such restriction is not necessary. Complex values of χ_e, which account for dissipation, are admissible here.

B. Ferrite Phase Shifter in Rectangular Waveguide

As the second example, consider the rectangular waveguide with a tranversely magnetized ferrite slab placed in the manner of Fig. 2. This problem falls in the category of inherent inhomogeneity with induced anisotropy [category 3)] and, in accordance with the previous remarks, λ is used as the parameter. Thus,

$$L(x, \nu) = \omega\begin{bmatrix} \epsilon_0\chi_e I & 0 \\ 0 & \mu_0\chi_m\end{bmatrix}; \qquad d \le x \le d+\delta \tag{37}$$

where χ_m, for the indicated orientation of the magnetic field, has the form

$$\chi_m = \begin{bmatrix} \chi_1 & 0 & -\chi_2 \\ 0 & 0 & 0 \\ \chi_2 & 0 & \chi_1\end{bmatrix}. \tag{38}$$

Because of the asymmetries of the susceptibility tensor and the geometric structure, the waveguide is nonreciprocal. The significant quantity of interest is the differential phase shift as defined by Lax, Button, and Roth [11], who have analyzed this problem exactly by numerical methods. To attack the problem with the perturbation method, we recall that the first-order perturbation of the wave number of the backward wave in a medium is identical to that of the forward wave in the transposed medium. Thus,

$$\Delta\kappa_{\alpha 1} = L_{-\alpha,\alpha} - \overset{T}{L}_{-\alpha,\alpha}. \tag{39}$$

It is again assumed that the $H_{1,0}$ mode is the only one present in the absence of the ferrite, so that (39) reduces to

$$\Delta\kappa_{(1,0)1} = \omega\mu_0 \int \boldsymbol{H}_{-(1,0)} \cdot (\chi_m - \chi_m{}^T) \cdot \boldsymbol{H}_{1,0} da. \tag{40}$$

Evaluation of the integral yields

$$\Delta\kappa_{(1,0)1} = \frac{2i\chi_2}{x_0} [\cos^2 k_{1,0}(\delta + d) - \cos^2 k_{1,0} d]. \tag{41}$$

For the sake of comparison, (41) is expanded in a Taylor series about $\delta = 0$ the first term of which is

$$\Delta\kappa_{(1,0)1} = -\frac{2i\chi_2}{x_0} k_{1,0}\delta \sin 2k_{1,0} d. \tag{42}$$

This expression differs from the first term of the expansion obtained from the exact transcendental eigenvalue equation [11] by the factor $(1+\chi_1)^{-1}$. The approximation of Lax, et al. [11], is valid for ratios of slab thickness to waveguide width that do not exceed one per cent. For greater accuracy, the higher order terms must be considered, but the evaluation of even the second-order term is quite involved and it is not included here in the interest of brevity.

V. Conclusion

It has been shown that the six-vector eigenvalue formulation of Maxwell's equations for guided waves enables the development of general perturbation and variational approximations in a concise and unified manner [13]. These approximations were shown to reduce to those given by Berk [7] for nondissipative media. A large number of problems of inhomogeneous, anisotropic, dissipative media can be treated in principle by these methods; but the accuracy of the results depends on the conditions in a particular configuration. From the practical viewpoint, the perturbation method is cumbersome if terms of order higher than the second are required. The method is generally more successful in problems which permit the expansion of the perturbation operator in a Taylor series as in (33).

Appendix

A convenient zero-order orthonormal set is constructed from the E and H modal solutions in cylindrical waveguides. These may be derived from Hertzian potentials of the E and H types [12]. For the H type:

$$\begin{bmatrix} \boldsymbol{E}_\alpha \\ i\boldsymbol{H}_\alpha \end{bmatrix} = \begin{bmatrix} i\omega\mu_0 \nabla \times \hat{z}\psi_{\alpha h} \\ i\nabla \times \nabla \times \hat{z}\psi_{\alpha h} \end{bmatrix}$$

$$(\nabla_t{}^2 + k_{\alpha h}{}^2)\psi_{\alpha h} = 0; \qquad \frac{\partial \psi_{\alpha h}}{\partial n} = 0 \text{ on boundary.}$$

The normalization condition requires that

$$\int \psi_{\alpha h}\psi_{\alpha h} da = -\frac{1}{2\omega\mu_0\kappa_{\alpha h}k_{\alpha h}{}^2}$$

where

$$k_{\alpha h}{}^2 = \omega^2\mu_0\epsilon_0 - \kappa_{\alpha h}{}^2.$$

For the E type:

$$\begin{bmatrix} \boldsymbol{E}_\alpha \\ i\boldsymbol{H}_\alpha \end{bmatrix} = \begin{bmatrix} \nabla \times \nabla \times \hat{z}\psi_{\alpha e} \\ \omega\epsilon_0 \nabla \times \hat{z}\psi_{\alpha e} \end{bmatrix}$$

$$(\nabla_t{}^2 + k_{\alpha e}{}^2)\psi_{\alpha e} = 0; \qquad \psi_{\alpha e} = 0 \text{ on boundary.}$$

The normalization condition requires that

$$\int \psi_{\alpha e}\psi_{\alpha e} da = -\frac{1}{2\omega\epsilon_0\kappa_{\alpha e}k_{\alpha e}{}^2}.$$

As an orthonormal set, the eigenfunctions are arranged in accordance with increasing wave number regardless of type.

References

[1] Van Trier, A. A., Guided electromagnetic waves in anisotropic media, *Appl. Sci. Res.*, vol B3, No 4 & 5, 1953, pp 305–371.
[2] Kales, M. L., Modes in waveguides containing ferrites, *J. Appl. Phys.*, vol 24, May, 1953, pp 604–608.
[3] Epstein, P. S., Theory of wave propagation in gyromagnetic medium, *Rev. Mod. Phys.*, vol 28, Jan 1956, pp 3–17.
[4] Suhl, H. and R. L. Walker, Topics in guided wave propagation through gyromagnetic media, part II, *Bell Sys. Tech. J.*, vol 33, Jul 1964, pp 939–986.
[5] Bresler, A. D., G. H. Joshi, and N. Marcuvitz, Orthogonality properties for modes in passive and active uniform waveguides, *J. Appl. Phys.*, vol 29, May, 1958, pp 794–799.
[6] Bresler, A. D., and N. Marcuvitz, Operator methods in electromagnetic field theory II, Res Rept R-565-57, Microwave Research Inst., Polytechnic Institute of Brooklyn, N. Y., March, 1956.
[7] Berk, A. D., Variational principles for electromagnetic resonators and waveguides, *IRE Trans. on Antennas and Propagation*, vol AP-4, Apr 1956, pp 104–111; and Berk, A. D., Tech Rept 284, M.I.T., Research Lab. of Electronics, Cambridge, Mass., Feb 1955.
[8] Villeneuve, A. T., Orthogonality relationships for waveguides and cavities with inhomogeneous anisotropic media, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-7, Oct 1959, pp 441–446.
[9] For further discussion of this point see: Schiff, L. I., *Quantum Mechanics*. New York: McGraw-Hill, 1955, ch VII, p 154.
[10] Morse, P. M., and H. Feshback, *Methods of Theoretical Physics*, vol II. New York: McGraw-Hill, 1953, ch 9, p 1108.
[11] Lax, B., K. J. Button, and L. Roth, The ferrite phase shifter in rectangular waveguide, *J. Appl. Phys.*, vol 25, November 1954, pp 1413–1421.
[12] Collin, R. E., *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960, ch V, p 170.
[13] It was indicated by one of the referees that equivalent expressions can be formulated using the reaction concept: Villeneuve, A. T., Reciprocity relations for gyrotropic media, Ph.D. dissertation, Syracuse University, N. Y., November, 1958.